Differential Geometry IV: General Relativity

12.1 Let us consider the following initial value problem on $(\mathbb{R}^{n+1}, \eta)$ (with $(t, x^1, \ldots, x^n)$ denoting the usual Cartesian coordinate system):

$$\begin{cases} \Box_\eta \psi = -(\partial_t \psi)^2, \\ \psi|_{t=0} = \psi_0, \quad \partial_t \psi|_{t=0} = \psi_1, \quad \text{for } \psi_0, \psi_1 \in C^\infty(\mathbb{R}^n). \end{cases} \tag{1}$$

(a) Show that, if the above initial value problem has a smooth solution on a domain of the form $\{-T_1 \leqslant t \leqslant T_2\}$, then it is unique. *(Hint: Consider the equation satisfied by the difference $w = \psi^{(1)} - \psi^{(2)}$ of two solutions; recast it as an equation which is linear in terms of w and apply an energy estimate.*

(b) Show that the initial value problem (1) satisfies the domain of dependence property: If $(\psi_0, \psi_1) = (0, 0)$ on a ball $B_{x_0,R} = \{x \in \mathbb{R}^n : |x - x_0| \leqslant R\}$, then ψ vanishes on the domain of dependence of $B_{x_0,R}$, i.e. on the diamond domain $\Omega = \{(t, x) : |x - x_0| \leqslant R - |t|\}$ (you can assume that any solution, if it exists, is sufficiently smooth; this is something that we will prove in class).

(c) Show that for any initial data of the form $(\psi_0, \psi_1) = (0, c)$, where $c \neq 0$ is a constant, the corresponding solution of (1) blows up in finite time (i.e. $\limsup_{t \to T^-} \|\psi(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} = +\infty$ for some $T \in \mathbb{R} \setminus 0$) *(Hint: Observe that, for a solution which is constant in x, the above equation becomes a second order ODE in terms of t).*

(d) In the case $n \geqslant 3$, show that, for any $\epsilon > 0$, there exist *compactly supported* initial data (ψ_0, ψ_1) with $\|\psi_0\|_{H^1} + \|\psi_1\|_{L^2} < \epsilon$ which blow up in finite time. As a result, small initial energy is not enough to guarantee that (1) has a solution which exists globally in time; this is related to the fact that (1) is *supercritical* with respect to the energy norm. *(Hint: Consider the initial data from part (c) and cut them off in a ball of suitable radius, using the domain of dependence property to deduce the behaviour of the solution on the domain of dependence of that ball.)*

Remark. In the case where $n = 3$ (that is, the physical space dimension), it was shown by Fritz John that *every* smooth and compactly supported initial data set (ψ_0, ψ_1) for (1) that doesn't vanish identically gives rise to a solution that blows up in finite time.

12.2 In this exercise, we will establish Hardy's inequality. Like Poincare's inequality, this inequality is used to control lower order norms of a function f in terms of norms of higher derivatives, albeit in unbounded domains.

(a) Let $f : [0, +\infty) \to \mathbb{R}$ be a C^1 function. Show that, for any $a > -1$, there exists a constant $C_a > 0$ depending only on a such that

$$\int_0^{+\infty} x^a |f|^2 \, dx \leqslant C_a \Big(\int_0^{+\infty} x^{a+2} |f'|^2 \, dx + \limsup_{x \to +\infty} (x^{a+1} |f|^2(x)) \Big).$$

In particular, if f is compactly supported, we have

$$\int_0^{+\infty} x^a |f|^2\, dx \leqslant C_a \int_0^{+\infty} x^{a+2} |f'|^2\, dx.$$

(Hint: Starting from the left hand side, write $x^a = \frac{d}{dx}\Big(\frac{1}{a+1}x^{a+1}\Big)$ *and integrate by parts; you might want to use a Cauchy–Schwarz inequality at some point.)*

(b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. Show that, for any $a > -n$, there exists a constant $C_{a,n} > 0$ depending only on a, n such that

$$\int_{\mathbb{R}^n} |x|^a |f|^2\, dx \leqslant C_{a,n}\Big(\int_{\mathbb{R}^n} |x|^{a+2} |\nabla f|^2\, dx + \limsup_{R\to+\infty} \int_{|x|=R} R^{a+1} |f|^2\, d\sigma \Big).$$

(Hint: Use polar coordinates.

(c) Let ψ be a smooth solution of the wave equation

$$\Box_\eta \psi = 0$$

on $(\mathbb{R}^n, \eta)$, $n \geqslant 3$, with smooth and compactly supported initial data (ψ_0, ψ_1) at $t = 0$. Show that

$$\sup_{T\in\mathbb{R}} \int_{t=T} \frac{1}{|x|^2} |\psi|^2\, dx \leqslant C\mathcal{E}[\psi](0) < +\infty$$

for some constant $C > 0$ depending only on the dimension n, where $E[\psi](0)$ is the initial energy of ψ. Compare the above bound with Exercise 10.2.b.

12.3 In this exercise, we will show that solutions of the standard wave equation on Minkowski spacetime decay polynomially fast towards timelike infinity. We will restrict ourselves to $(\mathbb{R}^{3+1}, \eta)$, even though similar arguments can be applied to the case of $\mathbb{R}^{n+1}$ for any $n \geqslant 3$. In what follows, we will denote with ϕ a given solution of

$$\Box_\eta \phi = 0$$

on $\mathbb{R}^{3+1}$ emanating from smooth and compactly supported initial data at $t = 0$. We will denote by $r = |x|$ the usual spatial radius function, and by

$$u = t - r, \quad v = t + r$$

the usual double null coordinate functions.

(a) Show that, in the polar coordinates (t, r, ω) on $\mathbb{R}^{3+1} \setminus \{r = 0\} \simeq \mathbb{R} \times (0, +\infty) \times \mathbb{S}^2$, the wave equation can be reexpressed in terms of $r\phi$ as

$$-\partial_t^2 (r\phi) + \partial_r^2 (r\phi) + \frac{1}{r^2} \Delta_{\mathbb{S}^2} (r\phi) = 0,$$

where $\delta_{\mathbb{S}^2}$ is the usual Laplace–Beltrami operator on the unit sphere $(\mathbb{S}^2, g_{\mathbb{S}^2})$; in the standard (θ, φ) coordinates on $\mathbb{S}^2$, it takes the form

$$\Delta_{\mathbb{S}^2} f = \frac{1}{\sin\theta}\partial_\theta\big(\sin\theta\partial_\theta f\big) + \frac{1}{\sin^2\theta}\partial^2_\varphi f.$$

In the (u, v, ω) coordinate system, we have

$$-4\partial_u\partial_v(r\phi) + \frac{1}{r^2}\Delta_{\mathbb{S}^2}(r\phi) = 0. \tag{2}$$

(b) Let us now switch to the (u, v, ω) coordinate system. Show the following energy bounds along the leaves of the foliation $\{u = const\}$: For any $0 \leqslant \tau_1 \leqslant \tau_2$,

$$\int_{u=\tau_2}\Big(4|\partial_v(r\phi)|^2 + \frac{1}{r^2}|\nabla_\omega(r\phi)|^2_{g_{\mathbb{S}^2}}\Big)\,dvd\omega \leqslant \int_{u=\tau_1}\Big(4|\partial_v(r\phi)|^2 + \frac{1}{r^2}|\nabla_\omega(r\phi)|^2_{g_{\mathbb{S}^2}}\Big)\,dvd\omega \tag{3}$$

and

$$\int_{u=\tau_1}\Big(4|\partial_v(r\phi)|^2+\frac{1}{r^2}|\nabla_\omega(r\phi)|^2_{g_{\mathbb{S}^2}}\Big)\,dvd\omega \leqslant \int_{t=0}\Big(|\partial_t(r\phi)|^2+|\partial_r(r\phi)|^2+\frac{1}{r^2}|\nabla_\omega(r\phi)|^2_{g_{\mathbb{S}^2}}\Big)\,drd\omega,$$

where $|\nabla_\omega(r\phi)|^2_{g_{\mathbb{S}^2}}$ denotes the norm of the spherical gradient, evaluated with respect to the unit metric on the sphere, while $d\omega$ denotes the associated volume form on the unit sphere. The above is of course nothing more than the usual energy inequality, applied to domains bounded by the null cones.

(Hint: Multiply (2) *with* $\partial_t(r\phi) = (\partial_u + \partial_v)(r\phi)$ *and integrate by parts over appropriate domains with respect to the volume form* $dudvd\omega$*. Note that you might have to truncate your domains by intersecting with* $\{v \leqslant V\}$ *and sending* $V \to +\infty$*.)*

(c) For any $p \in [0, 2]$, using $r^p\partial_v(r\phi)$ as a multiplier for (2), show that, for any $0 \leqslant \tau_1 \leqslant \tau_2$:

$$\int_{u=\tau_2} r^p|\partial_v(r\phi)|^2\,dvd\omega + \int_{\tau_1\leqslant u\leqslant\tau_2} r^{p-1}\Big(|\partial_v(r\phi)|^2 + r^{-2}|\nabla_\omega(r\phi)|^2_{g_{\mathbb{S}^2}}\Big)\,dudvd\omega \tag{4}$$
$$\leqslant \int_{u=\tau_2} r^p|\partial_v(r\phi)|^2\,dvd\omega.$$

and

$$\int_{u=0} r^p|\partial_v(r\phi)|^2\,dvd\omega \leqslant C\int_{t=0} r^p\Big(|\partial_v(r\phi)|^2 + r^{-2}|\nabla_\omega(r\phi)|^2_{g_{\mathbb{S}^2}}\Big)\,drd\omega$$

for some absolute constant $C > 0$ (i.e. independent of ϕ).

(d) Show that if $f : [0, +\infty) \to [0, +\infty)$ is a continuous function satisfying

$$\int_0^{+\infty} f(t)\,dt \leqslant C$$

then there exists a sequence of points $t_n \to +\infty$ with $t_{n+1} \in [2t_n, 4t_n]$ (we will call such a sequence *dyadic*) such that

$$f(t_n) \leqslant \frac{10C}{t_n}.$$

*(e) Starting from (4) for $p=2$ and using part (c), show that there exists a dyadic sequence τ_n and some absolute constant $C>0$, such that

$$\int_{u=\tau_n} r|\partial_v(r\phi)|^2\,dvd\omega \leqslant \frac{C}{\tau_n}\int_{t=0} r^2\Big(|\partial_v(r\phi)|^2 + r^{-2}|\nabla_\omega(r\phi)|^2_{g_{\mathbb{S}^2}}\Big)\,drd\omega.$$

Applying again (4) but this time for $p=1$ on the intervals $\{\tau_n \leqslant u \leqslant \tau_{n+1}$, show that, for some $\tau_n' \in [\tau_n, \tau_{n+1}]$:

$$\int_{u=\tau_n'} \big(|\partial_v(r\phi)|^2 + r^{-2}|\nabla_\omega(r\phi)|^2_{g_{\mathbb{S}^2}}\Big)\,dvd\omega \leqslant \frac{C'}{\tau_n^2}\int_{t=0} r^2\Big(|\partial_v(r\phi)|^2 + r^{-2}|\nabla_\omega(r\phi)|^2_{g_{\mathbb{S}^2}}\Big)\,drd\omega$$

for some absolute constant $C'>0$.

(f) Using the energy bound (3), deduce that, for any $\tau \geqslant 0$:

$$\int_{u=\tau} \big(|\partial_v(r\phi)|^2 + r^{-2}|\nabla_\omega(r\phi)|^2_{g_{\mathbb{S}^2}}\Big)\,dvd\omega \leqslant \frac{C''}{\tau^2}\int_{t=0} r^2\Big(|\partial_v(r\phi)|^2 + r^{-2}|\nabla_\omega(r\phi)|^2_{g_{\mathbb{S}^2}}\Big)\,drd\omega$$

for some absolute constant $C''>0$. In particular, the energy flux through the cones $\{u=\tau\}$ *decays* like τ^{-2} as $\tau\to+\infty$.

*(g) By commuting the wave equation with Cartesian derivatives, deduce an analogous decay estimate for higher order energy fluxes of ϕ. Applying the Sobolev inequality, deduce that

$$\phi(u,v,\omega) \lesssim u^{-1} \quad \text{and} \quad r\phi(u,v,\omega) \lesssim u^{-\frac{1}{2}},$$

with the constant implicit in the $\lesssim$ notation depending only on the initial data at $t=0$.